

# OPERATOR ALGEBRAS AND REPRESENTATIONS FROM COMMUTING SEMIGROUP ACTIONS

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**ABSTRACT.** Let  $\mathcal{S}$  be a countable, abelian semigroup of continuous surjections on a compact metric space  $X$ . Corresponding to this dynamical system we associate two operator algebras, the tensor algebra, and the semicrossed product. There is a unique smallest  $C^*$ -algebra into which an operator algebra is completely isometrically embedded, which is the  $C^*$ -envelope. We provide two distinct characterizations of the  $C^*$ -envelope of the tensor algebra; one developed in a general setting by Katsura, and the other using tools of projective and inductive limits, which gives the  $C^*$ -envelope as a crossed product  $C^*$ -algebra. We also study two natural classes of representations, the left regular representations and the orbit representations. The first is Shilov, and the second has a Shilov resolution.

## 1. INTRODUCTION

Let  $X$  be a compact metric space, and  $\mathcal{S}$  an abelian semigroup and let  $\sigma$  be a map of  $\mathcal{S}$  into the set of continuous, surjective maps of  $X \rightarrow X$ , which we assume to be a semigroup isomorphism. From this dynamical system  $(X, \sigma, \mathcal{S})$  we construct two operator algebras: the tensor algebra, and the semicrossed product.

If the semigroup  $\mathcal{S}$  is a group, then the tensor algebra and the semicrossed product coincide with the crossed product,  $C(X) \rtimes_{\sigma} \mathcal{S}$ . Our interest is in dealing with noninvertible dynamics, so we will assume that the semigroup  $\mathcal{S}$  is not a group.

Work on such problems began with single-variable dynamics [1], [13] and many others. Work with multivariate dynamics is more recent. This paper is in a sense a counterpoint to the important contribution of Davidson and Katsoulis [4], in which they studied various operator

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2000 *Mathematics Subject Classification.* 47D03 primary; 46H25, 20M14, 37B99 secondary.

*Key words and phrases.* abelian semigroup, dynamical system,  $C^*$ -envelope, tensor algebra, semicrossed product, Shilov representation.

<sup>‡</sup> The author acknowledges partial support from the National Science Foundation, DMS-0750986.

algebras that could be considered multivariate analogues of the (single variable) semicrossed product, and developed the dilation theory and isomorphism properties of these algebras. In [5], Donsig, Katavolos and Manoussos give precise description of the Jacobson radical of semicrossed products, where the semigroup is  $\mathbb{Z}_+^d$ .

While the point of view of  $C^*$ -dynamical systems mostly deals with group actions on  $C^*$ -algebras, Exel [6] and Exel and Renault [7] consider noninvertible dynamical systems, such as local homeomorphisms on a compact space. There is the additional feature of the transfer operator, which is not present here. Nevertheless it is interesting to compare their approach to the  $C^*$ -algebra which arises naturally in our context as the  $C^*$ -envelope of the tensor algebra.

We begin by constructing an algebra  $\mathcal{A}_0$  which contains operators  $S_s$  for  $s$  an element of the semigroup  $\mathcal{S}$ , and functions  $f \in C(X)$ , the continuous complex valued functions on  $X$ , subject to the covariance condition

$$f S_s = S_s f \circ \sigma_s.$$

An element of the algebra  $\mathcal{A}_0$  has the form  $\sum_s S_s f_x$ , where the sum is finite. We study classes of representations of this algebra. One natural class of representations which arises from the left regular representation the Hilbert space  $\ell_2(\mathcal{S})$  and the evaluation map of functions at a point  $x \in X$ . These representations, denoted by  $\pi$ , represent the operators  $S_s$  as isometries, and they separate the points of  $\mathcal{A}_0$ . Completing  $\mathcal{A}_0$  in the norm determined by these representations yields an algebra  $\mathcal{A}(X, \mathcal{S})$  which we call the left regular algebra. It turns out it is also the tensor algebra associated with a  $C^*$ -correspondence [10].

Another class of representations we study are the orbit representations. These are similar to the representations  $\pi$ , except they act on the orbit of a point  $x \in X$ . In the case of actions of a group  $G$ , the map  $G \rightarrow G \cdot x = \{g \cdot x : g \in G\}$  is bijective. But here, for non-invertible dynamics, the map  $\mathcal{S} \rightarrow \mathcal{S} \cdot x = \{\sigma_s(x) : s \in \mathcal{S}\}$  will not in general be one-to-one. (The fact we are assuming that the map  $\mathcal{S}$  into the semigroup of continuous surjective maps of  $X$  is a semigroup isomorphism does not imply that for a given  $x$ , the map  $s \in \mathcal{S} \rightarrow \sigma_s(x)$  is one-to-one.) We denote the orbit representations by  $\rho$ . These representations are thus a semigroup phenomenon not present when dealing with group actions. We show these representations are associated with cocycles, and indeed there is a one-to-one correspondence between the orbit representations and the orbit cocycles.

We have defined two nonselfadjoint operator algebras arising from the dynamical system  $(X, \sigma, \mathcal{S})$ . One is the tensor algebra, already mentioned. The other is the semicrossed product. This is the completion of the  $\mathcal{A}_0$  in the norm arising from considering all isometric covariant representations (Definition 2). However we have no tools to characterize all such representations, so there is little we can say about such algebras.

We present two constructions of the  $C^*$ -envelope of the tensor algebra  $\mathcal{A}(X, \mathcal{S})$ . The first, via  $C^*$ -correspondences, follows from work of Katsura [8], Muhly and Solel [10], and Davidson and Katsoulis [4]. A second approach to the  $C^*$ -envelope, which appears in [14] in the single variable setting, yields a more tangible result, yet is only available in a restricted context.

While the enveloping group  $\mathcal{G}$  containing the semigroup  $\mathcal{S}$ , is easily obtained as  $\mathcal{G} = \mathcal{S} - \mathcal{S}$ , there need not be any connection between the abstract group  $\mathcal{G}$  and mappings on the compact metric space  $X$ . In section 6 we construct a compact metric space  $\tilde{X}$  on which the group  $\mathcal{G}$  acts by homeomorphisms  $\tilde{\sigma}_g$  ( $g \in \mathcal{G}$ ) and a continuous surjection  $p : \tilde{X} \rightarrow X$  which “intertwines” this group action with the original semigroup action. Theorem 3 shows that the  $C^*$ -envelope of the tensor algebra  $\mathcal{A}(X, \mathcal{S})$  is identified with the  $C^*$ -crossed product  $C(\tilde{X}) \rtimes_{\tilde{\sigma}} \mathcal{G}$ .

The description of the  $C^*$ -envelope in Theorem 3 also yields some information about the left regular representations  $\pi$  and the (left regular) orbit representations  $\rho$ . We are able to show that the representations  $\pi$  are Shilov, and that the left regular orbit representations have a Shilov resolution.

We should comment on the relation of our results with those of [4]. They consider actions of the free semigroup on  $n$ -generators (for a fixed  $n \in \mathbb{N}$ ), whereas our semigroups are abelian. Even though we do not deal with specific examples of dynamical systems in this paper, it is also worth noting that there are actions which fall within our context which are not homomorphic images of free finitely generated semigroups: in [15], Example 5, there is an action of the semigroup of non-negative dyadic rationals on a compact metric space  $X$  by local homeomorphisms. We should also note that there is relatively little overlap of the results (with the exception of section 5). Because Davidson and Katsoulis deal with finitely many coordinates, they are able to obtain a number of dilation results. However the example of Parrott [11] of three commuting contractions which do not admit a unitary dilation illustrates the inherent difficulty of a general dilation theory in our

setting. What we are able to achieve, is a dilation of the commuting contractions  $\rho(S_s)$  ( $s \in \mathcal{S}$ ) to unitaries.

## 2. BACKGROUND AND NOTATION

**Standing Hypothesis** Throughout the paper,  $\mathcal{S}$  will denote an abelian semigroup with cancellation, and identity element, denoted by 0. The semigroup operation will be written as addition. The intersection of all abelian groups which contain  $\mathcal{S}$  will be written as  $\mathcal{G} = \mathcal{S} - \mathcal{S}$ .

While a few results do not require the commutativity of  $\mathcal{S}$ , nothing of importance is lost by making that assumption throughout the paper. We assume that  $\mathcal{S}$  acts on a compact metric space. Thus, there is a homomorphism, denoted by  $\sigma$ , from  $\mathcal{S}$  into the semigroup of continuous, surjective maps of  $X \rightarrow X$ . There is no loss of generality by assuming that  $\sigma$  is a semigroup isomorphism (onto its image), which we will do. Furthermore, we assume that  $\mathcal{S}$  is not a group, for otherwise nothing new is achieved. However, it may be the case that  $\mathcal{S}$  contains a non-trivial group  $\mathcal{S} \cap -\mathcal{S}$ . The triple  $(X, \sigma, \mathcal{S})$  will be called a dynamical system.

We will not keep repeating these assumptions in the statements of our results.

## 3. SEMICROSSED PRODUCTS

Let  $\mathcal{A}_0$  be the algebra generated by  $C(X)$  together with symbols  $S_s$ ,  $s \in \mathcal{S}$  and subject to the relations

$$\begin{aligned} fS_s &= S_sf \circ \sigma_s, & s \in \mathcal{S}, f \in C(X) \\ S_{s+t} &= S_sS_t, & s, t \in \mathcal{S} \end{aligned} \tag{*}$$

Thus a typical element of the algebra has the form

$$\sum_{s \in \mathcal{S}} S_sf_s$$

where the sum is finite.

Let  $\Gamma$  be the dual group of  $\mathcal{G}$ .

**Definition 1.** Define the *Gauge automorphism*  $\tau_\gamma$  ( $\gamma \in \Gamma$ ) on  $\mathcal{A}_0$  by

$$\tau_\gamma\left(\sum_s S_sf_s\right) = \sum_s \langle \gamma, s \rangle S_sf_s.$$

Define the projections  $P_s$ ,  $s \in \mathcal{S}$ : for  $F \in \mathcal{A}_0$ ,

$$P_s(F) = \int_\Gamma \tau_\gamma(F) \langle -\gamma, s \rangle d\gamma$$

where  $d\gamma$  is Haar measure on the compact group  $\Gamma$ . Note we are considering the semigroup  $\mathcal{S}$  and the group  $\mathcal{G}$  as discrete groups, and so  $\Gamma$  is a compact abelian group.

Note that if  $F = \sum_s S_s f_s \in \mathcal{A}_0$ , then  $P_{s_0}(F)$  is equal to either  $S_{s_0} f_{s_0}$  or 0 if  $s_0$  is not in the sum.

**Definition 2.** We say that a representation

$$\pi : \mathcal{A}_0 \rightarrow \mathcal{B}(\mathcal{H})$$

with the following properties:

- (1)  $\pi(S_s)$  is an isometry (resp., a contraction)  $\mathcal{B}(\mathcal{H})$  for all  $s \in \mathcal{S}$ ;
- (2)  $\pi(S_1) = I$  where 1 is the identity in  $\mathcal{S}$ ;
- (3)  $\pi|_{C(X)}$  is a  $C^*$ -representation.

is an isometric (resp., a contractive) covariant representation of the pair  $(C(X), \mathcal{S})$ .

In [4] Davidson and Katsoulis consider four sets of conditions on representations. But two of those conditions do not have a direct translation into this general context—namely, row contractive and row isometric, since our semigroup need not be freely generated by finitely many  $S_s$ .

**Definition 3.** Let  $C(X) \rtimes_{\sigma} \mathcal{S}$  denote the semicrossed product algebra; that is, the completion of  $\mathcal{A}_0$  with respect to the norm

$$\|F\| = \sup_{\pi} \|\pi(F)\|$$

for  $F \in \mathcal{A}_0$ , where the supremum is over all representations  $\pi$  satisfying properties (1), (2), (3) of the definition.

#### 4. THE LEFT REGULAR ALGEBRA

We now define a class of representations which will play an important role in what follows.

Given  $x \in X$  and  $\gamma \in \Gamma$  define a representation  $\pi_{x,\gamma}$  on the Hilbert space  $\ell_2(\mathcal{S})$  as follows: Let  $\xi_s \in \ell_2(\mathcal{S})$  be given by

$$\xi_s(t) = \begin{cases} 1 & \text{if } t = s; \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to define  $\pi_{x,\gamma}(f)$ ,  $f \in C(X)$ , and  $\pi_{x,\gamma}(S_t)$  on the vectors  $\xi_s$  since linear combinations of such vectors are dense. Set

$$\pi_{x,\gamma}(f)\xi_s = f \circ \sigma_s(x)\xi_s$$

and

$$\pi_{x,\gamma}(S_t)\xi_s = \langle \gamma, t \rangle \xi_{t+s}.$$

It is a routine calculation to verify that  $\pi_{x,\gamma}$  respects the relations  $\dagger$ . The adjoint is given by

$$\pi_{x,\gamma}(S_t)^*\xi_s = \begin{cases} \overline{\langle \gamma, t \rangle} \xi_u & \text{if } s = t + u \text{ for some } u \in \mathcal{S} \\ 0 & \text{otherwise.} \end{cases}$$

so that  $\pi_{x,\gamma}(S_t)^*\pi_{x,\gamma}(S_t)\xi_s = \xi_s$  for all  $s \in \mathcal{S}$ , and since the set  $\{\xi_s : s \in \mathcal{S}\}$  is an orthonormal basis for  $\ell_2(\mathcal{S})$ , it follows  $\pi_{x,\gamma}(S_t)^*\pi_{x,\gamma}(S_t) = I$ .

It is obvious that  $\pi_{x,\gamma}$  is a  $*$ -representation when restricted to  $C(X)$ . Thus it is an isometric covariant representation.

*Remark 1.* The representations  $\pi_{x,\gamma}$  in case  $\gamma$  is the trivial character are the semigroup analogue of the regular representations of crossed products, coming from the one dimensional evaluation representations, as in [16, 7.7].

**Lemma 1.** *Let  $F \in \mathcal{A}_0$ ,  $F \neq 0$ . Then for some  $x \in X$ ,  $\gamma \in \Gamma$ ,  $\pi_{x,\gamma}(F) \neq 0$ .*

*Proof.* Write  $F = \sum_{s \in I} S_s f_s$  where  $I$  is a finite subset of  $\mathcal{S}$ , and such that  $f_s \neq 0$  for  $s \in I$ . Let  $u \in \mathcal{S}$ ,  $s_0 \in I$  and compute

$$\begin{aligned} \int_{\Gamma} \pi_{x,\gamma}(F)\xi_u \langle -\gamma, s_0 \rangle d\gamma &= \sum_{s \in I} \int_{\Gamma} \pi_{x,\gamma}(S_s f_s)\xi_u \langle -\gamma, s_0 \rangle d\gamma \\ &= \sum_{s \in I} \int_{\Gamma} \langle \gamma, s \rangle \langle \gamma, -s_0 \rangle f_s(\sigma_u(x)) \xi_{s+u} d\gamma \\ &= f_{s_0}(\sigma_u(x)) \xi_{s+u} \end{aligned}$$

We may choose  $x \in X$  and  $u \in \mathcal{S}$  such that  $f_{s_0}(\sigma_u(x)) \neq 0$ . Thus, there is a choice of  $x \in X$  and  $\gamma \in \Gamma$  for which  $\pi_{x,\gamma}(F) \neq 0$ .  $\square$

**Corollary 1.** *The class of representations  $\pi_{x,\gamma}$ ,  $(x, \gamma) \in X \times \Gamma$  separates the elements of  $\mathcal{A}_0$ .*

**Definition 4.** We define the left regular algebra  $\mathcal{A}(\mathcal{S}, X)$  to be the completion of  $\mathcal{A}_0$  in the norm of the representation

$$\oplus_{(x,\gamma) \in X \times \Gamma} \pi_{x,\gamma}.$$

Let  $F \in \mathcal{A}(\mathcal{S}, X)$ ,  $\|F\| = 1$ , and suppose that for all  $u \in \mathcal{S}$ ,  $P_u(F) = 0$ . Now there are  $x \in X$ ,  $\gamma \in \Gamma$  and unit vectors  $\xi, \eta \in \mathcal{H}_x$  for which

$$|(\pi_{x,\gamma}(F)\xi, \eta)| > \frac{1}{2}\|F\|.$$

Hence there exist  $s, t \in \mathcal{S}$  such that

$$\epsilon := |(\pi_{x,\gamma}(F)\xi_s, \xi_t)| > 0.$$

Let  $G \in \mathcal{A}_0$  be such that  $\|F - G\| < \delta$ , where  $0 < \delta < \epsilon/2$ . Then

$$\begin{aligned} |(\pi_{x,\gamma}(G)\xi_s, \xi_t)| &\geq |(\pi_{x,\gamma}(F)\xi_s, \xi_t)| - \|F - G\| \\ &\geq \epsilon - \delta \\ &> \epsilon/2. \end{aligned}$$

Express  $G = \sum_u S_u f_u$ . Now  $|(\pi_{x,\gamma}(G)\xi_s, \xi_t)| > 0$  implies for  $u = t - s \in \mathcal{S}$ ,  $f_u \neq 0$ . Thus for  $u = t - s$  we have

$$P_u(G - F) = P_u(G) = S_u f_u.$$

Now

$$|(\pi_{x,\gamma}(G)\xi_s, \xi_t)| = |f_u(\sigma_s(x))| > \epsilon/2,$$

so that  $\|P_u(G)\| > \epsilon/2$ . On the other hand,

$$\|P_u(G)\| = \|P_u(F - G)\| \leq \|F - G\| < \epsilon/2.$$

We have shown the following:

**Proposition 1.** *If  $F$  is a nonzero element of  $\mathcal{A}(\mathcal{S}, X)$  then there exists  $u \in \mathcal{S}$  such that  $P_u(F) \neq 0$ .*

Notice also the following.

**Proposition 2.**  *$P_0$  is a faithful, completely contractive conditional expectation of  $\mathcal{A}(\mathcal{S}, X) \rightarrow C(X)$ .*

*Proof.* For  $F \in \mathcal{A}_0$ ,  $F = \sum f_s S_s$  (finite sum),  $P_0(F) = f_0$  where  $S_0 = I$ . So it is evident that

$$P_0(fF) = fP_0(F) = P_0(Ff)$$

for any  $f \in C(X)$ .

We see that for  $F \in \mathcal{A}_0$  as above,

$$P_0(F^*F) = \sum |f_s|^2$$

and in particular,

$$(\ddagger) \quad P_0(F^*F) \geq |f_s|^2 = P_s(F)^*P_s(F)$$

for any  $s$ . So, by continuity of  $P_0$  and density of  $\mathcal{A}_0$ , it follows that  $\ddagger$  holds for  $F \in \mathcal{A}(\mathcal{S}, X)$ .

Now suppose  $F \in \mathcal{A}(\mathcal{S}, X)$  and  $P_0(F^*F) = 0$ . Then it follows that  $P_s(F) = 0$  for all  $s \in \mathcal{S}$ . So, by the previous Proposition,  $F = 0$ .

The map  $F \in \mathcal{A}(X, \mathcal{S}) \rightarrow \tau_\gamma(F)$  is completely isometric. As  $P_0$  is the average of completely isometric maps, it is completely contractive.  $\square$

**Corollary 2.** *There is a faithful conditional expectation of the semi-crossed product  $C(X) \rtimes_\sigma \mathcal{S}$  onto  $C(X)$ .*

*Proof.* By the definition of the norm on the semicrossed product, there is a contractive map of  $C(X) \rtimes_{\sigma} \mathcal{S}$  onto  $\mathcal{A}(\mathcal{S}, X)$ . The composition of this map with the conditional expectation on  $\mathcal{A}(\mathcal{S}, X)$  is the desired expectation.  $\square$

**4.1. Orbit Representations.** Next we define another class of representations of  $\mathcal{A}_0$ , which we call orbit representations. Fix  $x \in X$  and let  $\mathcal{S}(x)$  denote the orbit of  $x$ , namely,  $\mathcal{S}(x) = \{\sigma_s(x) : s \in \mathcal{S}\}$ .

**Definition 5.** A function  $\mu : \mathcal{S} \times \mathcal{S}(x) \rightarrow \mathbb{C}$  is an *orbit cocycle* if it satisfies

(1) For each  $t \in \mathcal{S}$  any  $y \in \mathcal{S}(x)$

$$\sum_{\sigma_t(y_j) = \sigma_t(y)} |\mu(t, y_j)|^2 \leq 1$$

(2) (cocycle condition) For each  $s, t \in \mathcal{S}$ , and  $y \in \mathcal{S}(x)$

$$\mu(s + t, y) = \mu(t, y) \mu(s, \sigma_t(y)).$$

We may also write  $\mu_x$  to emphasize the dependence on the point  $x \in X$ .

We will define the orbit representations  $\rho_{x, \mu}$ .

The Hilbert space is  $\ell_2(\mathcal{S}(x))$ . For  $y \in \mathcal{S}(x)$ , let  $\xi_y$  be the function

$$\xi_y(w) = \begin{cases} 1, & \text{if } w = y \\ 0, & \text{otherwise.} \end{cases}$$

Define, for  $f \in C(X)$ ,  $y \in \mathcal{S}(x)$

$$\rho_{x, \mu}(f) \xi_y = f(y) \xi_y$$

and

$$\rho_{x, \mu}(S_t) \xi_y = \mu(t, y) \xi_{\sigma_t(y)}.$$

Let  $\xi = \sum a_i \xi_{y_i}$  be a unit vector in  $\ell_2(\mathcal{S}(x))$  such that  $\sigma_t(y_i) = \sigma_t(y)$ .

$$\rho_{x, \mu}(S_t) \xi = \left( \sum a_j \mu(t, y_j) \right) \xi_{\sigma_t(y)}.$$

Hence

$$\begin{aligned} \|\rho_{x, \mu}(S_t) \xi\|^2 &= \left| \sum a_j \mu(t, y_j) \right|^2 \\ &\leq \left( \sum |a_j|^2 \right) \left( \sum |\mu(t, y_j)|^2 \right). \end{aligned}$$

Since  $\xi$  is a unit vector,  $\sum |a_j|^2 = 1$ . Hence, if  $\rho_{x, \mu}(S_t)$  is to be contractive, we must have that  $\sum |\mu(t, y_j)|^2 \leq 1$ . On the other hand, let us note that this condition is sufficient for  $\rho_{x, \mu}(S_t)$  to be contractive.



Consider the dense set of vectors which are linear combinations of vectors  $\xi$  of the above form. Say  $\eta = \sum b_k \xi_k$ , where for each  $k$ ,  $\rho_{x,\mu}(S_t)\xi_k$  is a multiple of  $\xi_{u_k}$  for some  $u_k \in \mathcal{S}$ , where the  $u_k$  are distinct elements of  $\mathcal{S}$ , the  $\xi_k$  are unit vectors, and  $\sum |b_k|^2 = 1$ . Then by the above we have that

$$\begin{aligned} \|\rho_{x,\mu}(S_t)\eta\|^2 &= \left\| \sum \rho_{x,\mu}(S_t)b_k \xi_k \right\|^2 \\ &\leq \left\| \sum b_j \xi_{u_k} \right\|^2 \\ &\leq 1 \end{aligned}$$

Additionally we have, for  $s, t \in \mathcal{S}$  and  $y \in \mathcal{S}(x)$

$$\begin{aligned} \rho_{x,\mu}(S_{s+t})\xi_y &= \rho_{x,\mu}(S_s S_t)\xi_y \\ &= \rho_{x,\mu}(S_s)\rho_{x,\mu}(S_t)\xi_y \\ &= \rho_{x,\mu}(S_s)\mu(t, y)\xi_{\sigma_t(y)} \\ &= \mu(t, y)\mu(s, \sigma_t(y))\xi_{\sigma_{s+t}(y)} \\ &= \mu(s + t, y)\xi_{\sigma_{s+t}(y)} \end{aligned}$$

To conclude that  $\rho_{x,\mu}$  is a representation, we need  $\rho_{x,\mu}(fS_s) = \rho_{x,\mu}(S_s f \circ \sigma_s)$ ,  $s \in \mathcal{S}$ ,  $f \in C(X)$ . But that is a routine calculation.

We summarize this as

**Corollary 3.** *Orbit representations are contractive covariant representations. Furthermore, there is a one-to-one correspondence between orbit representations and orbit cocycles.*

*Remark 2.* Let  $\mu$  be an orbit cocycle, and  $\gamma \in \Gamma$ . Then  $\gamma\mu$  is also an orbit cocycle. That is,  $\gamma\mu(t, y) = \langle \gamma, t \rangle \mu(t, y)$ .

To address the question of what can be said about the existence of orbit cocycles we need a definition from [15]

**Definition 6.** A *cocycle* for a dynamical system  $(X, \sigma, \mathcal{S})$  is a map  $\omega : \mathcal{S} \times X \rightarrow \mathbb{R}$  such that

- (1)  $\omega(s, x) \geq 0$  for all  $s \in \mathcal{S}$ ,  $x \in X$ ;
- (2) For each  $y \in X$ ,  $t \in \mathcal{S}$ ,  $\sum_{\sigma_t(x)=y} \omega(t, x) = 1$ ;
- (3) For each  $t \in \mathcal{S}$ , the map  $x \rightarrow \omega(t, x)$  is continuous;
- (4) For each  $s, t \in \mathcal{S}$  and  $x \in X$ ,  $\omega$  satisfies the cocycle identity

$$\omega(s + t, x) = \omega(s, x) \omega(t, \sigma_s(x)).$$

If the dynamical system  $(X, \sigma, \mathcal{S})$  admits a cocycle, then given  $x \in X$  one can define an orbit cocycle  $\mu_x$  by letting  $\mu_x(t, y) = \sqrt{\omega(t, \sigma_t(y))}$ , for  $t \in \mathcal{S}$  and  $y$  in the orbit of  $x$ .

*Example 1.* [15] considers abelian semigroup actions on a compact metric space by continuous, surjective, locally injective maps. Proposition 2 gives necessary and sufficient conditions for a  $\mathbb{Z}_+^k$  actions to admit a cocycle, and Example 5 of [15] is an action of the non-negative dyadic rationals on a compact metric space by local homeomorphisms which admits a cocycle.

Let us see how the two classes of representations  $\pi_{x,\gamma}$  and  $\rho_{x,\mu}$  are related. Let  $x \in X$  be fixed. Define an equivalence relation  $\sim$  on the semigroup  $\mathcal{S}$  by  $s \sim t$  if  $\sigma_s(x) = \sigma_t(x)$ , and let  $[s]$  denote an equivalence class. Define a map  $q : \mathcal{S} \rightarrow \mathcal{S}(x)$  by  $q(s) = \sigma_s(x)$ . Then  $q$  is a one-to-one surjective map of the set of equivalence classes  $\mathcal{S}/\sim$  to the orbit  $\mathcal{S}(x)$ .

Let  $\mathcal{H}_0^0$  be the subspace of all (finite) linear combinations  $\eta = \sum a_s \xi_s$  for which  $\sum a_s \xi_{q(s)} = 0$ . Note that any such sum is the sum of elements  $\sum a_s \xi_{q(s)}$  for which the  $s$  appearing in the sum belong to the same equivalence class, and the sum of the coefficients  $\sum a_s = 0$ .

**Lemma 2.** *The linear space  $\mathcal{H}_0^0$  is invariant under the maps  $\pi_{x,\gamma}(f)$ ,  $f \in C(X)$ , and under  $\pi_{x,\gamma}(S_t)$ ,  $t \in \mathcal{S}$ . Hence the closure of  $\mathcal{H}_0^0$ , which we denote by  $\mathcal{H}_0$ , is invariant under  $\pi_{x,\gamma}(F)$ , for  $F \in \mathcal{A}_0$ ,  $\gamma \in \Gamma$ .*

*Proof.* Let  $f \in C(X)$  and  $\eta \in \mathcal{H}_0^0$ ,  $\eta = \sum a_j \xi_{s_j}$  where the  $s_j$  belong to the same equivalence class, say  $s_j \in [s]$ , and  $\sum a_j = 0$ . Then

$$\begin{aligned} \pi_{x,\gamma}(f)\eta &= \sum f(\sigma_{s_j}(x)) a_j \xi_{s_j} \\ &= \sum f(\sigma_s(x)) a_j \xi_{s_j} \\ &\in \mathcal{H}_0^0 \end{aligned}$$

For  $t \in \mathcal{S}$ ,

$$\begin{aligned} \pi_{x,\gamma}(S_t)\eta &= \sum \langle \gamma, t \rangle a_j \xi_{t+s_j} \\ &\in \mathcal{H}_0^0 \end{aligned}$$

because if  $s_j$  belong to the same equivalence class, then so do the elements  $s_j + t$ , since

$$\sigma_{s_j+t}(x) = \sigma_t \circ \sigma_{s_j}(x) = \sigma_t(\sigma_{s_j}(x))$$

where  $[s_j] = [s]$  for all  $j$ . □

Let  $Q$  denote the orthogonal projection of  $\ell_2(\mathcal{S})$  onto the subspace  $\mathcal{H}_0$ . Let  $\mathcal{H}_1 = Q^\perp(\ell_2(\mathcal{S}))$ .

Since  $\mathcal{H}_0$  is invariant, we can define the representation  $\pi_{x,\gamma}^0$  to be the restriction of  $\pi_{x,\gamma}$  to the subspace  $\mathcal{H}_0$ . The subspace  $\mathcal{H}_1$  need not be

invariant, but we can define the representation  $\pi_{x,\gamma}^1$  by

$$\pi_{x,\gamma}^1(F) = Q^\perp \pi_{x,\gamma}(F)|\mathcal{H}_1.$$

Note that  $Q^\perp \xi_s \neq 0$  for all  $s \in \mathcal{S}$ .

For simplicity of notation, if  $\gamma = 1$  is the trivial character, write  $\pi_{x,1} = \pi_x$ ,  $\pi_{x,1}^1 = \pi_x^1$ .

**Definition 7.** Define an orbit cocycle  $\mu$  by setting, for  $y \in \mathcal{S}(x)$ ,  $t \in \mathcal{S}$ ,

$$\mu(t, y) = \frac{\|\pi_x^1(S_t)Q^\perp \xi_u\|}{\|Q^\perp \xi_u\|} = \frac{\|Q^\perp \xi_{t+u}\|}{\|Q^\perp \xi_u\|}$$

if  $y = \sigma_u(x)$ . This is well-defined, for if  $y = \sigma_{u'}(x)$ , then  $Q^\perp \xi_u = Q^\perp \xi_{u'}$ . We call  $\mu$  the *left regular orbit cocycle*.

**Lemma 3.**  $\mu$  satisfies the two conditions of Definition 5, and hence is an orbit cocycle. Furthermore,  $\mu(s, y) \neq 0$  for all  $s \in \mathcal{S}$  and  $y \in \mathcal{S}(x)$ .

*Proof.* Let  $s, t \in \mathcal{S}$  and  $y \in \mathcal{S}(x)$ ,  $y = \sigma_u(x)$ . Then

$$\begin{aligned} \mu(t, y) \mu(s, \sigma_t(y)) &= \frac{\|Q^\perp \xi_{t+u}\|}{\|Q^\perp \xi_u\|} \frac{\|Q^\perp \xi_{s+t+u}\|}{\|Q^\perp \xi_{t+u}\|} \\ &= \frac{\|Q^\perp \xi_{s+t+u}\|}{\|Q^\perp \xi_u\|} \\ &= \mu(s+t, y), \end{aligned}$$

verifying the cocycle identity.

Suppose  $u_j$ ,  $j = 1, \dots, n$  are elements of  $\mathcal{S}$  such that if  $y_j = \sigma_{u_j}(x) \in \mathcal{S}(x)$  are distinct, and  $\sigma_t(y_j) = \sigma_t(y)$ ,  $1 \leq j \leq n$ , where  $y = y_1 = \sigma_u(x)$  and  $u = u_1$ .

The vectors  $U_j = \frac{1}{\|Q^\perp \xi_{u_j}\|} Q^\perp \xi_{u_j}$  are mutually orthogonal unit vectors, and  $\xi = \sum a_j U_j$  is a unit vector if  $a_j \in \mathbb{C}$  satisfy  $\sum_{j=1}^n |a_j|^2 = 1$ . Now

$$\begin{aligned} \pi_x^1(S_t)\xi &= \left( \sum \frac{a_j}{\|Q^\perp \xi_{u_j}\|} \right) Q^\perp \xi_{t+u} \\ &= \left( \sum a_j \frac{\|Q^\perp \xi_{t+u}\|}{\|Q^\perp \xi_{u_j}\|} \right) \frac{1}{\|Q^\perp \xi_{t+u}\|} Q^\perp \xi_{t+u} \end{aligned}$$

Since  $\pi_x^1(S_t)$  is contractive,  $\|\pi_x^1(S_t)\xi\| \leq 1$ . Hence, the scalar  $|\sum a_j \frac{\|Q^\perp \xi_{t+u}\|}{\|Q^\perp \xi_{u_j}\|}| \leq 1$ , for all choices of  $a_j$  such that  $\sum_{j=1}^n |a_j|^2 = 1$ . By Cauchy-Schwarz, this implies that

$$\sum \left( \frac{\|Q^\perp \xi_{t+u}\|}{\|Q^\perp \xi_{u_j}\|} \right)^2 \leq 1.$$

In other words,

$$\sum_j \mu(t, y_j)^2 \leq 1.$$

Finally,  $\mu$  is never zero since  $Q^\perp \xi_s \neq 0$  for all  $s \in \mathcal{S}$ . □

With  $\mu$  the left regular orbit cocycle, define  $W : \ell_2(\mathcal{S}(x)) \rightarrow \mathcal{H}_1$  as follows: if  $y \in \mathcal{S}(x)$ , say  $y = \sigma_s(x)$ , set  $W\xi_y = \frac{1}{\|Q^\perp \xi_s\|} Q^\perp \xi_s$ . Then  $W$  maps an orthonormal basis of  $\ell_2(\mathcal{S}(x))$  onto an orthonormal basis of  $\mathcal{H}^1$ . We compute

$$\begin{aligned} W^* \pi_{x,\gamma}^1(S_t) W \xi_y &= W^* \pi_{x,\gamma}^1(S_t) \frac{1}{\|Q^\perp \xi_s\|} Q^\perp \xi_s \\ &= \langle \gamma, t \rangle \frac{1}{\|Q^\perp \xi_s\|} W^* Q^\perp \xi_{t+s} \\ &= \langle \gamma, t \rangle \frac{\|Q^\perp \xi_{t+s}\|}{\|Q^\perp \xi_s\|} W^* \frac{1}{\|Q^\perp \xi_{t+s}\|} \xi_{t+s} \\ &= \mu(t, y) \rho_{x,\gamma\mu}(S_t) \xi_{\sigma_t}(y) \end{aligned}$$

Also, a straightforward calculation shows that  $W^* \pi_{x,\gamma}^1(f) W \xi_y = \rho_{x,\gamma\mu}(f) \xi_y$ . This proves

**Corollary 4.**  $W^* \pi_{x,\gamma}^1(F) W = \rho_{x,\gamma\mu}(F)$ , where  $F \in \mathcal{A}_0$ ,  $\gamma \in \Gamma$ , and  $\mu$  is the left regular orbit cocycle. Thus,

$$\|\rho_{x,\gamma\mu}(F)\| \leq \|\pi_{x,\gamma}(F)\|.$$

**Lemma 4.** For each  $x \in X$ , let  $\mu_x$  be the left regular orbit cocycle. Then for some  $x \in X$ ,  $\gamma \in \Gamma$ ,  $\rho_{x,\gamma\mu_x}(F) \neq 0$ .

*Proof.* The proof is similar to that of Lemma 1. Let  $F \in \mathcal{A}$ ,  $F \neq 0$  and  $I$  a finite subset of  $\mathcal{S}$ , and such that  $f_s \neq 0$  for  $s \in I$ . Let  $s_0 \in I$  and  $x \in X$  such that  $f_{s_0}(x) \neq 0$ . compute

Integrate  $\rho_{x,\gamma\mu_x}(F) \xi_x \langle \gamma, s_0 \rangle$  with respect to the Haar measure on  $\Gamma$ . This yields  $\mu_x(s_0, x) f_{s_0}(x) \xi_{\sigma_{s_0}}(x) \neq 0$ . Thus there is some  $\gamma \in \Gamma$  for which  $\rho_{x,\gamma\mu_x}(F) \neq 0$ . □

*Remark 3.* In Definition 4 we defined the left regular norm on  $\mathcal{A}_0$  as the supremum over the norms of the representations  $\pi_{x,\gamma}$ . In light of Lemma 4, one could define a norm on  $\mathcal{A}_0$  by taking the supremum of the norms of the left regular orbit representations. That norm is evidently dominated by the left regular norm. It would certainly be interesting to determine under what conditions the two norms agree.

5. THE TENSOR ALGEBRA AND  $C^*$ -CORRESPONDENCES

In this section we show that the left regular algebra  $\mathcal{A}(X, \mathcal{S})$  can be identified with the tensor algebra of a  $C^*$ -correspondence as developed by Muhly-Solel [10] and Katsura [8]. Our context is similar to that of [4], and hence is abbreviated.

We will view  $\mathcal{E}_0 = C_{00}(\mathcal{S}, C(X))$ , the vector space of finitely supported functions from the semigroup  $\mathcal{S}$  to  $C(X)$ , as a pre-Hilbert  $C^*$ -module over  $C(X)$ .  $C(X)$  acts on the right as a pointwise product. Thus if  $\xi \in \mathcal{E}_0$ ,  $(\xi \cdot f)$  is the function given by  $(\xi \cdot f)(s) = \xi(s)f$ .

$\mathcal{E}_0$  is endowed with a  $C(X)$ -valued sesquilinear form  $\langle \cdot, \cdot \rangle \rightarrow C(X)$  satisfying

- (1)  $\langle \cdot, \cdot \rangle$  is conjugate linear in the first variable, so  $\langle \xi \cdot f, \eta \cdot g \rangle = f^* \langle \xi, \eta \rangle g$ ,  $\xi, \eta \in \mathcal{E}_0$ ,  $f, g \in C(X)$ .
- (2) For all  $\xi \in \mathcal{E}_0$ ,  $\langle \xi, \xi \rangle$  is a positive element of  $C(X)$  which is zero if and only if  $\xi = 0$ .

The inner product is defined by:

$$\langle \xi, \eta \rangle(x) = \sum_s \overline{\xi(s)(x)} \eta(s)(x)$$

where the sum is finite. A norm is then given by  $\|\xi\| = \sqrt{\|\langle \xi, \xi \rangle\|}$ . We let  $\mathcal{E}$  be the completion of  $\mathcal{E}_0$  in the given norm.

The elements of  $\mathcal{E}$  are functions supported on the set  $E = X \times \mathcal{S}$ . We define a left action of  $C(X)$  on  $\mathcal{E}$  by

$$(f \cdot \xi)(x, s) = \varphi(f)\xi(x, s) = \xi(x, s)(f \circ \sigma_s)(x).$$

Thus,  $\mathcal{E}$  is a (right) Hilbert  $C^*$ -module over  $C(X)$  with the additional structure as a left module over  $C(X)$ .

The adjointable left multipliers of  $\mathcal{E}$  form a  $C^*$ -algebra,  $\mathcal{L}(\mathcal{E})$ . We can identify the left action of  $C(X)$  into  $\mathcal{L}(\mathcal{E})$  by

$$\varphi(f) = \text{diag}(f \circ \sigma_s)_{s \in \mathcal{S}}.$$

We will write  $\varphi(f)\xi$  rather than  $f \cdot \xi$  to be clear that the left action is not multiplication of functions.

Because the maps  $\sigma_s$ ,  $s \in \mathcal{S}$  are assumed to be surjective,  $\varphi$  is faithful. The  $C^*$ -module is full, i.e.,  $\langle \mathcal{E} | \mathcal{E} \rangle = C(X)$ . Also, the left action is essential:  $\varphi(C(X))\mathcal{E} = \mathcal{E}$ .

We review the construction of the tensor algebra of  $\mathcal{E}$  following Muhly and Solel. Set  $\mathcal{E}^{\otimes 0} = C(X)$  and

$$\mathcal{E}^{\otimes k} = \underbrace{\mathcal{E} \otimes_{C(X)} \mathcal{E} \otimes_{C(X)} \cdots \mathcal{E}}_{k \text{ copies}} \text{ for } k \geq 1.$$

Notice that  $\xi f \otimes \eta = \xi \otimes \varphi(f)\eta$ .

Let  $\epsilon_s$  denote the “column vector” with a 1 in the  $s$  position. That is,  $\epsilon_s$  is the element of  $\mathcal{E}$  which maps the point  $s \in \mathcal{S}$  to the constant function  $\mathbf{1}$ , and maps  $t \neq s \in \mathcal{S}$  to the function  $\mathbf{0}$ . The  $\{\epsilon_s\}_{s \in \mathcal{S}}$  form an orthormal family of unit vectors (i.e.,  $\langle \epsilon_s, \epsilon_t \rangle = \mathbf{1}\delta_{s,t}$ .) Furthermore one can show that for any  $\xi \in \mathcal{E}$ ,  $\xi = \sum_{s \in \mathcal{S}} \epsilon_s \cdot \langle \epsilon_s, \xi \rangle$ , so that the family is an orthonormal basis of the Hilbert  $C^*$ -module.

Given  $s_k, \dots, s_1 \in \mathcal{S}$  let  $w$  be the word  $w = (s_k, \dots, s_1)$  and write  $\epsilon_w = \epsilon_{s_k} \otimes \dots \otimes \epsilon_{s_1}$ . A typical element of  $\mathcal{E}^{\otimes k}$  has the form  $\sum_{|w|=k} \epsilon_w f_w$  for  $f_w \in C(X)$ .

$\mathcal{E}^{\otimes k}$  is naturally a  $C(X)$  bimodule with

$$(\xi_k \otimes \dots \otimes \xi_1) \cdot f = \xi_k \otimes \dots \otimes (\xi_1 f)$$

and

$$f \cdot (\xi_k \otimes \dots \otimes \xi_1) = (\varphi(f)\xi_k) \otimes \dots \otimes \xi_1.$$

the  $C(X)$ -valued inner product is given on basis elements by

$$\langle \epsilon_w f_w, \epsilon_v g_v \rangle = \overline{f_w} g_v \text{ and } \langle \epsilon_w f_w, \epsilon_v g_v \rangle = 0$$

if  $v \neq w$ .

The Fock space of  $\mathcal{E}$  is  $\mathcal{F}(\mathcal{E}) = \sum_{n \geq 0}^{\oplus} \mathcal{E}^{\otimes n}$ , which becomes a  $C^*$ -correspondence with the same actions as defined on each summand, and the  $C(X)$  inner product obtained by declaring the distinct summands to be orthogonal. One obtains a  $*$ -isomorphism  $\varphi_\infty$  of  $C(X)$  into  $\mathcal{L}(\mathcal{F}(\mathcal{E}))$  by letting  $\varphi_\infty(f)$  act as  $\varphi_k(f)$  on each summand.

The operator  $T_\xi^{(k)}$  for  $\xi \in \mathcal{E}$  maps  $\mathcal{E}^{\otimes k}$  into  $\mathcal{E}^{\otimes k+1}$  by

$$T_\xi^{(k)}(\xi_1 \otimes \dots \otimes \xi_k) = \xi \otimes \xi_1 \otimes \dots \otimes \xi_k.$$

$T_\xi$  is the operator on  $\mathcal{F}(\mathcal{E})$  which restricts to  $T_\xi^{(k)}$  on  $\mathcal{E}^{\otimes k}$ . The tensor algebra  $\mathcal{T}_+(\mathcal{E})$  of  $\mathcal{E}$  is the norm-closed nonself-adjoint subalgebra generated by  $\varphi_\infty(C(X))$  and  $\{T_\xi : \xi \in \mathcal{E}\}$ . The  $C^*$ -algebra generated by  $\mathcal{T}_+(\mathcal{E})$  is the Toeplitz  $C^*$ -algebra  $\mathcal{T}(\mathcal{E})$ .

A representation of a  $C^*$ -correspondence  $\mathcal{E}$  consists of a linear map  $\Lambda$  of  $\mathcal{E}$  into  $\mathcal{B}(\mathcal{H})$  and a representation  $\lambda$  of  $C(X)$  on  $\mathcal{H}$  such that

- (1)  $\Lambda(\xi)^* \Lambda(\eta) = \lambda(\langle \xi | \eta \rangle)$  for all  $\xi, \eta \in \mathcal{E}$
- (2)  $\lambda(f) \Lambda(\xi) = \Lambda(\varphi(f)\xi)$  for all  $f \in C(X), \xi \in \mathcal{E}$ .

Let  $C^*(\Lambda, \lambda)$  be the  $C^*$ -algebra generated  $\lambda(C(X))$  and  $\Lambda(\mathcal{E})$ .

We next consider a class of representations of  $\mathcal{E}$ . Choose  $x \in X$  and let  $\mathcal{H} = \ell_2(\mathcal{S})$ . It is enough to define  $\Lambda$  on the basis elements  $\epsilon_s f$ ,  $f \in$

$C(X)$ . Here we revert to the previous notation  $\xi_t$  for the function in  $\ell_2(\mathcal{S})$  which is 1 at  $s = t$  and 0 elsewhere. Define a representation by

$$\Lambda_{x,\gamma}(\epsilon_s f)\xi_t = \langle \gamma, s \rangle f \circ \sigma_t(x) \xi_{s+t}, \text{ and } \lambda_{\gamma,x}(f)\xi_t = f(\sigma_t(x)).$$

Set  $\Lambda = \bigoplus_{(\gamma,x) \in \Gamma \times X} \Lambda_{x,\gamma}$  and  $\lambda = \bigoplus_{(\gamma,x) \in \Gamma \times X} \lambda_{x,\gamma}$ .

There is an action of the circle  $\mathbb{T}$  on  $C^*(\Lambda, \lambda)$  defined as in [4] prior to Theorem 5.1. This “gauge action” is not to be confused with the gauge automorphism of Definition 1.

**Theorem 1.**  $\mathcal{A}(X, \mathcal{S})$  is completely isomorphic to the tensor algebra  $\mathcal{T}_+(\mathcal{E})$ . Furthermore, the  $C^*$ -envelope of  $\mathcal{A}(X, \mathcal{S})$  is  $\mathcal{O}(E)$ , the Cuntz-Pimsner algebra.

*Proof.* Let  $\xi, \eta \in \mathcal{E}$ , say  $\xi = \sum_s \epsilon_s f_s$ ,  $\eta = \sum_s \epsilon_s g_s$ , where the sums are finite. Then

$$\bigoplus_{(x,\gamma) \in X \times \Gamma} \Lambda_{x,\gamma}(\xi) \Lambda_{x,\gamma}(\eta)^* = \bigoplus_{(x,\gamma) \in X \times \Gamma} \sum_s \pi_{x,\gamma}(S_s) \pi_{x,\gamma}(f_s) \pi_{x,\gamma}(g_s)^* \pi_{x,\gamma}(S_s)^*$$

and as noted in section 4,  $\pi_{x,\gamma}(S_t)^*$  annihilates vectors  $\xi_t \in \ell_2(\mathcal{S})$  for which  $t - s \notin \mathcal{S}$ . On the other hand, there is no  $0 \neq h \in C(X)$  such that  $\lambda(h)$  annihilates a unit vector in the Hilbert space of  $\lambda$ . Thus, as observed in [4], Theorem 6.2 of [8] shows that we obtain a faithful representation of  $\mathcal{T}_{\mathcal{E}}$ , the universal  $C^*$ -algebra generated by all representations of the  $C^*$ -correspondence, which is shown to be isomorphic to  $\mathcal{T}(\mathcal{E})$ .  $\square$

## 6. EXTENSIONS OF SEMIGROUP DYNAMICAL SYSTEMS

**Definition 8.** Given a dynamical system  $(X, \sigma, \mathcal{S})$  we say that the dynamical system  $(Y, \beta, \mathcal{S})$  is an extension of  $(X, \sigma, \mathcal{S})$  if there is a continuous surjection  $p : Y \rightarrow X$  such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\beta_s} & Y \\ p \downarrow & & p \downarrow \\ X & \xrightarrow{\sigma_s} & X \end{array}$$

commutes for every  $s \in \mathcal{S}$ . We call  $p$  the extension map of  $(Y, \beta, \mathcal{S})$  over  $(X, \sigma, \mathcal{S})$ .

We say that an extension  $(Y, \beta, \mathcal{S})$  is a *homeomorphism extension* of  $(X, \sigma, \mathcal{S})$  if the maps  $\beta_s$  are homeomorphisms for all  $s \in \mathcal{S}$ . We now provide a procedure for producing a canonical homeomorphism extension of  $(X, \sigma, \mathcal{S})$ .

Let  $\mathcal{G} = \mathcal{S} - \mathcal{S}$  be the group generated by the abelian semigroup  $\mathcal{S}$ . (Recall that  $\mathcal{S}$  is a semigroup with cancellation.) Define a partial order

on  $\mathcal{G}$  by  $h < g$  if  $g - h \in \mathcal{S}$ . Let  $X_g = X$  for all  $g \in \mathcal{G}$ . If  $g - h = u \in \mathcal{S}$  let  $\sigma_u$  map  $X_g \rightarrow X_h$ . Then the commutativity conditions for an inverse system are satisfied, so the inverse limit (or projective limit) of the inverse system exists. Denote the inverse limit by  $\tilde{X}$ .

**Proposition 3.**  $\tilde{X} = \{(x_g)_{g \in \mathcal{G}} \in \prod X_g : x_h = \sigma_u(x_g) \text{ for all } h < g \in \mathcal{G}, \text{ with } u = g - h\}$ .

*Proof.* This is [2, Proposition 16-6.4].  $\square$

We now show that there is a homeomorphism  $\tilde{\sigma}_t$ , for each  $t \in \mathcal{S}$ . Let  $\tilde{\sigma}_t$  be the map  $\tilde{\sigma}_t(x_g)_{g \in \mathcal{G}} = (\sigma_t(x_s))_{s \in \mathcal{G}}$ , and let  $p : \tilde{X} \rightarrow X$  be the map  $p((x_s)_{s \in \mathcal{G}}) = x_0$  (where 0 is the identity of  $\mathcal{G}$ .)

**Proposition 4.**  $(\tilde{X}, \tilde{\sigma}, \mathcal{S})$  is a dynamical system for which the  $\tilde{\sigma}_t$  are homeomorphisms, for all  $t \in \mathcal{S}$ . Furthermore, the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\sigma}_t} & \tilde{X} \\ p \downarrow & & p \downarrow \\ X & \xrightarrow{\sigma_t} & X \end{array}$$

commutes, so that  $(\tilde{X}, \tilde{\sigma}, \mathcal{S})$  is a homeomorphism extension of  $(X, \sigma, \mathcal{S})$ .

*Proof.* We first see that  $\tilde{\sigma}_t$  is surjective. Indeed, let  $((y_g)_{g \in \mathcal{G}}) \in \tilde{X}$ , and set  $x_g = y_{g+t}$ . Then  $(x_g)_{g \in \mathcal{G}} \in \tilde{X}$ , and  $\tilde{\sigma}_t((x_g)_{g \in \mathcal{G}}) = (y_g)_{g \in \mathcal{G}}$ .

To show injectivity suppose  $(x_g)_{g \in \mathcal{G}}, (x'_g)_{g \in \mathcal{G}} \in \tilde{X}$  and  $\tilde{\sigma}_t((x_g)_{g \in \mathcal{G}}) = \tilde{\sigma}_t((x'_g)_{g \in \mathcal{G}})$ . Then for all  $g \in \mathcal{G}$ ,  $x_{g-t} = x'_{g-t}$ . Hence  $(x_g)_{g \in \mathcal{G}} = (x'_g)_{g \in \mathcal{G}}$ .  $\square$

**Corollary 5.**  $\mathcal{G}$  acts as a group of homeomorphisms on  $\tilde{X}$ .

*Proof.* Let  $g \in \mathcal{G}$ . Since  $\mathcal{S} - \mathcal{S} = \mathcal{G}$ ,  $g$  can be written as  $s - t$ , for  $s, t \in \mathcal{S}$ . Define

$$\tilde{\sigma}_g = \tilde{\sigma}_s \circ \tilde{\sigma}_t^{-1}.$$

We show this is well defined. If also  $g = s' - t'$ , then  $s + t' = s' + t$ . Hence,  $\tilde{\sigma}_{s+t'} = \tilde{\sigma}_{s'+t}$ . From this we obtain  $\tilde{\sigma}_s \circ \tilde{\sigma}_t^{-1} = \tilde{\sigma}_{s'} \circ \tilde{\sigma}_{t'}^{-1}$ .  $\square$

Our next goal is to show that the extension  $(\tilde{X}, \tilde{\sigma}, \mathcal{S})$  is a minimal extension of  $(X, \sigma, \mathcal{S})$  in a sense we will make precise.

**Lemma 5.** If  $\sigma_t$  is a homeomorphism for all  $t \in \mathcal{S}$ , then the map  $p : \tilde{X} \rightarrow X$  is a homeomorphism. Hence the dynamical systems  $(X, \sigma, \mathcal{S})$  and  $(\tilde{X}, \tilde{\sigma}, \mathcal{S})$  are conjugate.



*Proof.* We need only show that the map  $p$  is injective since By Definition 8 it is a continuous surjection. So assume that  $p((x_s)_{s \in \mathcal{S}}) = p((y_s)_{s \in \mathcal{S}})$ . In particular  $x_0 = y_0$ . Now since  $\sigma_t$  is a homeomorphism we have that  $x_s = \sigma_s^{-1}(x_0) = \sigma_s^{-1}(y_0) = y_s$  and hence  $p$  is injective. That the systems are conjugate follows from the commutative diagram for the notion of extension.  $\square$

**Definition 9.** Consider an extension  $(Y, \beta, \mathcal{S})$  of  $(X, \sigma, \mathcal{S})$  via an extension map  $r$ . We say that an extension  $(Z, \varphi, \mathcal{S})$  of  $(X, \sigma, \mathcal{S})$  lies between  $(Y, \beta, \mathcal{S})$  and  $(X, \sigma, \mathcal{S})$  if the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\beta_t} & Y \\ p \downarrow & & p \downarrow \\ Z & \xrightarrow{\varphi_t} & Z \\ q \downarrow & & q \downarrow \\ X & \xrightarrow{\sigma_t} & X \end{array}$$

commutes for all  $t \in \mathcal{S}$ , and  $q \circ p = r$ , where  $p$  and  $q$  are the extension maps as in the diagram.

We say the extension  $(Y, \beta, \mathcal{S})$  of  $(X, \sigma, \mathcal{S})$  via an extension map  $r$  is a *homeomorphism extension* if the maps  $\beta_t$ ,  $t \in \mathcal{S}$  are homeomorphisms, for  $t \in \mathcal{S}$ . Finally, we call a homeomorphism extension  $(Y, \beta, \mathcal{S})$  of  $(X, \sigma, \mathcal{S})$  *minimal* if for any dynamical system  $(Z, \varphi, \mathcal{S})$  that lies between the two systems as in the diagram, the extension map  $p$  is a homeomorphism, and hence  $(Y, \beta, \mathcal{S})$  and  $(Z, \varphi, \mathcal{S})$  are conjugate systems.

We refer to the homeomorphism extension  $(\tilde{X}, \sigma, \mathcal{S})$  of  $(X, \sigma, \mathcal{S})$  as the *canonical* homeomorphism extension.

**Lemma 6.** *The canonical homeomorphism extension of  $(X, \sigma, \mathcal{S})$  is a minimal extension.*

*Proof.* Assume that we have the following commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\sigma}_t} & \tilde{X} \\ p \downarrow & & p \downarrow \\ Z & \xrightarrow{\varphi_t} & Z \\ q \downarrow & & q \downarrow \\ X & \xrightarrow{\sigma_t} & X \end{array}$$

where  $\tilde{\sigma}_t$  and  $\varphi_t$  are homeomorphisms for every  $t$  and  $p$  and  $q$  are surjections with  $q \circ p((x_s)_{s \in \mathcal{S}}) = x_1$ . By the preceding lemma we know that  $\tilde{Z}$  and  $Z$  are conjugate and hence we will show that  $\tilde{Z}$  and  $\tilde{X}$  are conjugate.

For notational purposes we will interchange the notations  $x = (x_s)_{s \in \mathcal{S}}$  for an element of  $\tilde{X}$  as necessary. We define a map  $\Gamma : \tilde{X} \rightarrow \tilde{Z}$  by  $\Gamma((x_s)_{s \in \mathcal{S}}) = (\varphi_s^{-1}(p(x)))_{s \in \mathcal{S}}$ . The map  $\Gamma$  is clearly continuous. On the other hand notice that  $\varphi_t(\varphi_{st}^{-1}(p(x))) = \varphi_s^{-1}(p(x))$  and hence  $\Gamma(x) \in \tilde{Z}$ .

Now if  $(y_s)_{s \in \mathcal{S}} \in \tilde{Z}$  then there exists  $x \in \tilde{X}$  such that  $p(x) = y_1$  since  $p$  is surjective. Also  $y_s = \varphi_s^{-1}(y_1)$  and hence  $\Gamma(x) = (y_s)_{s \in \mathcal{S}}$  and hence  $\Gamma$  is onto.

To see that  $\Gamma$  is one-to-one consider the map  $\Pi : \tilde{Z} \rightarrow \tilde{X}$  given by  $\Pi((y_s)_{s \in \mathcal{S}}) = (q(y_s))_{s \in \mathcal{S}}$ . Notice that  $\varphi_t(q(y_{st})) = q(\varphi_t(y_{st})) = q(y_s)$  and so the map  $\Pi$  does map into  $\tilde{X}$ . Now we see that

$$\begin{aligned} \Pi \circ \Gamma(x) &= (q(\varphi_s^{-1}(p(x))))_{s \in \mathcal{S}} \\ &= (q(p \circ \tilde{\sigma}_s^{-1}(x)))_{s \in \mathcal{S}} \\ &= (q \circ p((x_{ts})_{t \in \mathcal{S}}))_{s \in \mathcal{S}} \\ &= (x_s)_{s \in \mathcal{S}} = x. \end{aligned}$$

It follows that  $\Gamma$  is one-to-one and hence  $\Gamma$  is a homeomorphism. The conjugacy follows immediately from the commutative diagram.  $\square$

The next theorem now follows immediately.

**Theorem 2.** *The dynamical system  $(X, \sigma, \mathcal{S})$  has a minimal homeomorphism extension, which is unique up to conjugacy.*

**6.1. Dualizing the canonical homeomorphism extension.** Let  $\mathcal{A} = C(X)$ , and  $\alpha_s$  be the endomorphism  $\alpha_s(f) = f \circ \sigma_s$ ,  $s \in \mathcal{S}$ ,  $f \in C(X)$ . Define a partial order on  $\mathcal{S}$  by  $t \succ s$  if there exists  $u \in \mathcal{S}$  such that  $t = u + s$ . Now the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha_s} & \mathcal{A} \\ \alpha_t \downarrow & & \alpha_u \downarrow \\ \mathcal{A} & \xlongequal{\quad} & \mathcal{A} \end{array}$$

commutes, so we can form the inductive system  $(\mathcal{A}, \alpha_s)$  with respect to the order  $\succ$ . Let  $\tilde{\mathcal{A}} = \varinjlim (\mathcal{A}_s, \alpha_s)$  where  $\mathcal{A}_s = \mathcal{A}$  for all  $s$ , and let  $\iota_s$  be the canonical embeddings of  $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ . Thus we have the commutative

diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha_s} & \mathcal{A} \\ \downarrow \iota_{t+s} & & \downarrow \iota_t \\ \tilde{\mathcal{A}} & \xlongequal{\quad} & \tilde{\mathcal{A}} \end{array}$$

Now we define  $\tilde{\alpha}_s : \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$  as follows: if  $\tilde{a} \in \tilde{\mathcal{A}}$  there exists a  $t \in \mathcal{S}$  and  $a \in \mathcal{A}$  such that  $\tilde{a} = \iota_t(a)$ . Define

$$\tilde{\alpha}_s(\tilde{a}) = \iota_t(\alpha_s(a)).$$

This is well defined by the commutativity of the diagrams. Since  $\alpha_s$ ,  $s \in \mathcal{S}$  is linear and injective, the same is true for  $\tilde{\alpha}_s$ . We show  $\tilde{\alpha}_s$  is invertible.

Let  $\tilde{a} \in \tilde{\mathcal{A}}$  be given; say  $\tilde{a} = \iota_t(a)$  for some  $t \in \mathcal{S}$ ,  $a \in \mathcal{A}$ . We can assume  $t \succ s$ , say  $t = u + s$  for some  $u \in \mathcal{S}$ . Then

$$\tilde{a} = \iota_t(a) = \iota_u \circ \alpha_s(b)$$

for some  $b \in \mathcal{A}$ . Thus,  $\tilde{a} = \tilde{\alpha}_s(\tilde{b})$  where  $\tilde{b} = \iota_u(b)$ .

Now the mappings  $\alpha_s$ ,  $\iota_t$ , ( $s, t \in \mathcal{S}$ ) are isometric and  $*$ -maps (i.e.  $\alpha_s(\bar{a}) = \overline{\alpha_s(a)}$ ) hence  $\tilde{\mathcal{A}}$  is the direct limit of  $C^*$ -algebras, so that the completion of  $\tilde{\mathcal{A}}$  is a commutative  $C^*$ -algebra,  $C(Z)$ . The automorphisms  $\tilde{\alpha}_s$  are isometric on  $\tilde{\mathcal{A}}$ , hence extend to automorphisms, also denoted  $\tilde{\alpha}_s$ , of  $C(Z)$ . Thus, by the Banach-Stone Theorem, there is a homeomorphism  $\varphi_s$  of  $Z$  such that  $\tilde{\alpha}_s(f) = f \circ \varphi_s$ ,  $f \in C(Z)$ ,  $s \in \mathcal{S}$ .

Let  $j$  be the embedding  $C(X) \hookrightarrow C(\tilde{X})$  given by  $j(f) = f \circ p$  where  $p : \tilde{X} \rightarrow X$  is the canonical map. Now for  $s \in \mathcal{S}$  let  $\beta_s : \mathcal{A} \rightarrow C(\tilde{X})$  be the map  $\beta_s(f) = j(f) \circ \tilde{\sigma}_{-s}$ . Then the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha_u} & \mathcal{A} \\ \downarrow \beta_s & & \downarrow \beta_{s+u} \\ C(\tilde{X}) & \xlongequal{\quad} & C(\tilde{X}) \end{array}$$

commutes. Thus, by properties of direct limits, there is a star homomorphism  $\Psi : \tilde{\mathcal{A}} \rightarrow C(\tilde{X})$ . Since the maps  $\beta_s$  are isometric, so is  $\Psi$ , hence  $\Psi$  extends to a map (also denoted  $\Psi$ ) of  $C(Z) \rightarrow C(\tilde{X})$ .

Now the embedding  $C(Z) \rightarrow C(\tilde{X})$  yields a map  $p : \tilde{X} \rightarrow Z$  as follows: let  $\tilde{x}$  be a pure state on  $C(\tilde{X})$ , which we identify with a point of  $\tilde{X}$ . Restricting  $\tilde{x}|_{C(Z)}$  yields a pure state of  $C(Z)$ , which is canonically identified with a point of  $Z$ .

We observe that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\sigma}_s} & \tilde{X} \\ p \downarrow & & p \downarrow \\ Z & \xrightarrow{\varphi_s} & Z \end{array}$$

commutes. By the minimal extension property of  $\tilde{X}$  (cf Lemma 6),  $p$  is a homeomorphism. Thus,  $C(\tilde{X})$  is the (completion of) the direct limit of the directed system  $(C(X), \alpha_s)$ .

## 7. THE $C^*$ -ENVELOPE REVISITED

**Theorem 3.** *The  $C^*$ -envelope of the left regular algebra  $\mathcal{A}(X, \mathcal{S})$  is the crossed product  $C(\tilde{X}) \rtimes_{\tilde{\alpha}} \mathcal{G}$ .*

*Proof.* Define representations  $\tilde{\pi}_{\hat{x}, \gamma}$  for  $x \in X$  and  $\gamma \in \Gamma$  of the crossed product  $C(\tilde{X}) \rtimes_{\tilde{\alpha}} \mathcal{G}$  as follows. Let  $\hat{x}$  be the subset  $p^{-1}(x) \subset \tilde{X}$ , where  $p$  is the map  $\tilde{X} \rightarrow X$  given in Definition 4. The Hilbert space is  $\ell_2(\mathcal{G}(\hat{x}))$ , where  $\mathcal{G}(\hat{x})$  denotes the union of the orbits  $\mathcal{G}(\tilde{x})$  for  $\tilde{x} \in \hat{x}$ . If  $U_g$  is the unitary element in the crossed product associated with the homeomorphism  $\tilde{\sigma}_g$ , the representation is given by

$$\tilde{\pi}_{\hat{x}, \gamma}(U_g)\xi_{\tilde{x}} = \langle \gamma, g \rangle \xi_{\tilde{\sigma}(x)}$$

where  $\xi_{\tilde{x}}$  is the function in  $\ell_2(\mathcal{G}(\hat{x}))$  which is 1 at  $\tilde{x}$  and zero elsewhere. And for  $\tilde{f} \in C(\tilde{X})$ ,  $\tilde{\pi}_{\hat{x}, \gamma}(\tilde{f})\xi_{\tilde{x}} = \tilde{f}(\tilde{x})\xi_{\tilde{x}}$ .

Since the direct sum of the representations  $\tilde{f} \rightarrow \tilde{f}|_{\hat{x}}$  (that is, the restriction of  $\tilde{f}$  to the subset  $\hat{x} \subset \tilde{X}$ ) of  $C(\tilde{X})$  is faithful, it follows that the supremum of the norms of the representations  $\tilde{\pi}_{\hat{x}, \gamma}$  is faithful on the crossed product, since  $\mathcal{G}$  is abelian, hence amenable. (cf [16, 7.7.5]) Indeed, this holds even if  $\gamma$  is taken to be the trivial character.

Since  $C(X)$  is embedded in  $C(\tilde{X})$  by the map  $j(f) = f \circ p$ , it follows that  $\tilde{\pi}_{\hat{x}, \gamma}(j(f))\xi_{\tilde{y}} = f(y)\xi_{\tilde{y}}$  for  $\tilde{y} \in \mathcal{S}(\tilde{x})$  with  $p(\tilde{y}) = y \in X$ , since  $j(f)$  is constant on the subset  $\hat{y} \subset \tilde{X}$ , and that constant is  $f(y)$ .

Let  $F \in \mathcal{A}_0$ , say  $F = \sum S_s f_s$  (where the sum is finite), let  $\tilde{F} = \sum U_s j(f_s)$ . Then we have that

$$\|\tilde{\pi}_{\hat{x}, \gamma}(\tilde{F})\| = \|\pi_{x, \gamma}(F)\|.$$

It follows that equality holds for  $F \in \mathcal{A}(X, \mathcal{S})$  and hence that the embedding of the left regular algebra  $\mathcal{A}(X, \mathcal{S})$  into the crossed product is completely isometric. We will also denote this embedding by  $j$ , which is consistent if we view  $C(X)$  as a subalgebra of  $\mathcal{A}(X, \mathcal{S})$  and  $C(\tilde{X})$  as a subalgebra of the crossed product.

To complete the proof, suppose  $\mathcal{B}$  is the  $C^*$ -envelope of  $\mathcal{A}(X, \mathcal{S})$ , and let  $k : \mathcal{A}(X, \mathcal{S}) \rightarrow \mathcal{B}$  be the completely isometric embedding. Then there is surjective  $C^*$ -homomorphism  $\Phi : C(\tilde{X}) \rtimes_{\tilde{\alpha}} \mathcal{G} \rightarrow \mathcal{B}$  such that the diagram

$$\begin{array}{ccc} \mathcal{A}(X, \mathcal{S}) & \xrightarrow{j} & C(\tilde{X}) \rtimes_{\tilde{\alpha}} \mathcal{G} \\ id \downarrow & & \downarrow \Phi \\ \mathcal{A}(X, \mathcal{S}) & \xrightarrow{k} & \mathcal{B} \end{array}$$

commutes. It remains to show that  $\Phi$  is an isomorphism. Suppose, to the contrary, there is an element  $H \in \ker(\Phi)$ . We may suppose  $H$  has norm 1.  $H$  can be approximated by an element  $G$  with  $\|G - H\| < \frac{1}{4}$ , where  $G = \sum_{i=1}^n U_{g_i} h_i$  with  $h_i \in C(\tilde{X})$ .

From Section 6.1 there exist  $f_i \in C(X)$  such that  $\|j(f_i) - h_i\| < \frac{1}{4n}$ ,  $1 \leq i \leq n$ . Thus if  $F = \sum U_{g_i} j(f_i)$ , then  $\|F - G\| < \frac{1}{4}$ .

Now express  $g_i = s_i - t_i$ , where  $s_i, t_i \in \mathcal{S}$ ,  $1 \leq i \leq n$ . Let  $U = U_{t_1} \cdots U_{t_n} = U_{t_1 \cdots t_n}$ . Then  $FU \in j(\mathcal{A}(X, \mathcal{S}))$ , and since  $U$  is unitary in the crossed product,  $\|FU\| = \|F\|$ .

Now  $\|H - F\| < \frac{1}{2}$ , so that  $\|\Psi(H - F)\| < \frac{1}{2}$ . Since  $\|H\| = 1$ , this implies  $\|F\| = \|FU\| > \frac{1}{2}$ . Hence,

$$\begin{aligned} \|\Psi(HU - FU)\| &\leq \|\Psi(H - F)\| \|\Psi(U)\| \\ &\leq \|\Psi(H - F)\| \\ &< \frac{1}{2} \end{aligned}$$

whereas, since  $\Psi(H) = 0$ ,

$$\begin{aligned} \|\Psi(HU - FU)\| &= \|\Psi(FU)\| \\ &> \frac{1}{2} \end{aligned}$$

since, by the defining property of the  $C^*$ -envelope,  $\Psi$  is completely isometric on  $j(\mathcal{A}(X, \mathcal{S}))$ . This contradiction shows that the kernel of  $\Psi$  is trivial, and so the crossed product is the  $C^*$ -envelope.  $\square$

**Corollary 6.** *Given  $x \in X$ ,  $\gamma \in \Gamma$ ,*

- (1) *the semigroup  $\pi_{x,\gamma}(S_s)$  ( $s \in \mathcal{S}$ ) of commuting isometries dilates to a commuting semigroup of unitaries;*
- (2) *the semigroup  $\rho_{x,\gamma}(S_s)$  ( $s \in \mathcal{S}$ ) of commuting contractions dilates to a commuting semigroup of unitaries.*

**Theorem 4.** *There is a completely contractive representation  $\Pi : C(X) \rtimes_{\sigma} \mathcal{S} \rightarrow \mathcal{A}(X, \mathcal{S})$ .*

*Proof.* Let  $F \in \mathcal{A}_0$ . The norm of  $F$  as an element of the semicrossed product  $C(X) \rtimes_{\sigma} \mathcal{S}$  is given as the supremum over all representations  $\|\pi(F)\|$  which satisfy the three properties of Definition 2. The norm of  $F$  as an element of the left regular algebra  $\mathcal{A}(X, \mathcal{S})$  is given as the supremum over a subset of these representations. Since the semicrossed product is the completion of  $\mathcal{A}_0$  in the larger norm, for  $F \in \mathcal{A}(X, \mathcal{S})$ , we may take

$$\Pi(F) = \oplus_{(x,\gamma) \in X \times \Gamma} \pi_{x,\gamma}(F).$$

This yields a contractive map of the semicrossed product into  $\mathcal{A}(X, \mathcal{S})$ .

To see the map is completely contractive, the proof of Theorem 3 shows that the representation  $\pi_{x,\gamma}(F)$  is unitarily equivalent to the restriction of  $\tilde{\pi}_{\hat{x},\gamma}(F)$  to an invariant subspace. Since this is a  $C^*$ -representation, it is completely contractive, and the same is true of the direct sum of such representations. Thus the map  $\Pi$  is completely contractive.  $\square$

*Remark 4.* If the map  $\Pi$  is not completely isometric, then it would be interesting to have examples of representations  $\pi$  for which the norm  $\|\pi(F)\|$  is not dominated by the norm of  $F$  in  $\mathcal{A}(X, \mathcal{S})$ . Conceivably such representations could be orbit representations for which the associated orbit cocycle is not the left regular orbit cocycle. Of course, the existence of such cocycles will depend on the semigroup  $\mathcal{S}$ . For, say if  $\mathcal{S} = \mathbb{N}$ , then the semicrossed product norm and the left regular norm coincide. More generally, what condition on the dynamical system  $\mathcal{S}$  is needed to insure that the two norms are different?

**Corollary 7.** *The conditional expectation given in Corollary 2 of  $C(X) \rtimes_{\sigma} \mathcal{S} \rightarrow C(X)$  is completely contractive.*

*Proof.* By Theorem 4 the map  $\Pi$  of  $C(X) \rtimes_{\sigma} \mathcal{S}$  onto  $\mathcal{A}(X, \mathcal{S})$  is completely contractive, and by Proposition 2 the conditional expectation of  $\mathcal{A}(X, \mathcal{S})$  onto  $C(X)$  is completely contractive. The conditional expectation on the semicrossed product is the composition of the two maps.  $\square$

### 7.1. Shilov modules.

**Definition 10.** Let  $\mathcal{A}$  be an operator algebra,  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  a representation. Then  $\pi$  is said to be a *Shilov* representation if there is a representation  $\Pi$  of the  $C^*$ -envelope  $C^*(\mathcal{A})$  in a Hilbert space  $\mathcal{K}$  containing  $\mathcal{H}$  as a subspace, so that (viewing  $\mathcal{A}$  as a subalgebra of  $C^*(\mathcal{A})$ ),  $\pi(F)$  is the restriction of  $\Pi(F)$  to  $\mathcal{H}$ , for all  $F \in \mathcal{A}$ . [9] expresses this in the language of modules:  $\mathcal{H}$  is isomorphic to a submodule of  $\mathcal{K}$  viewed as an  $\mathcal{A}$ -module.

A Hilbert module  $\mathcal{H}$  is said to have a *Shilov resolution* if there is a short exact sequence of  $\mathcal{A}$  modules

$$0 \rightarrow \mathcal{K}_0 \rightarrow \mathcal{K} \xrightarrow{\Phi} \mathcal{H} \rightarrow 0$$

where  $\mathcal{K}_0$  and  $\mathcal{K}$  are Shilov modules.

Let  $\mathcal{H}_0$ ,  $\pi_{x,\gamma}^1$ ,  $\mathcal{H}_1$ ,  $\pi_{x,\gamma}^1$  be the Hilbert spaces and representations introduced prior to Definition 7. While these were initially defined as representations of  $\mathcal{A}_0$ , they are uniquely extendible to representations of  $\mathcal{A} = \mathcal{A}(X, \mathcal{S})$ , and it is in this context we consider them here. Following [9], we employ the language of Hilbert modules.

Let us fix  $x \in X$  and  $\gamma \in \Gamma$ . View  $\mathcal{H}_0$  as an  $\mathcal{A}$  module via the representation  $\pi_{x,\gamma}^0$ ,  $\mathcal{H}_1$  as an  $\mathcal{A}$  module via the representation  $\pi_{x,\gamma}^1$ , and  $\ell_2(\mathcal{S})$  as an  $\mathcal{A}$  module via the representation  $\pi_{x,\gamma}$ .

**Theorem 5.** (1)  $\ell_2(\mathcal{S})$  is a Shilov module;  
 (2)  $\mathcal{H}_1$  has a Shilov resolution

$$0 \rightarrow \mathcal{H}_0 \rightarrow \ell_2(\mathcal{S}) \xrightarrow{Q^\perp} \mathcal{H}_1 \rightarrow 0.$$

*Proof.* (1) Theorem 3 shows that for  $F \in \mathcal{A}$ ,  $\pi_{x,\gamma}(F)$  is unitarily equivalent to the restriction of the representation  $\tilde{\pi}_{\hat{x},\gamma}(F)$  of the  $C^*$ -envelope to an invariant subspace.

(2) Since  $\pi_{x,\gamma}$  is a Shilov representation of  $\mathcal{A}$ , so is its restriction to an invariant subspace. Thus  $\mathcal{H}_0$  is a Shilov module. Since  $\mathcal{H}_1$  is the quotient space  $\ell_2(\mathcal{S})/\mathcal{H}_0$ , it has a Shilov resolution as given in (2).  $\square$

*Remark 5.* While the construction of the  $C^*$ -envelope in Theorem 1 via  $C^*$ -correspondences seems to suggest that the left regular representations  $\pi_{x,\gamma}$  will be Shilov, nevertheless it seems that we need the more concrete description of the  $C^*$ -envelope given by Theorem 3 to achieve that.

Again fixing  $x \in X$  and  $\gamma \in \Gamma$ , and let  $\mu = \mu_x$  be the left regular orbit cocycle, and  $\rho_{x,\gamma\mu}$  the associated representation of the orbit space  $\ell_2(\mathcal{S}(x))$ , which we view as an  $\mathcal{A}(X, \mathcal{S})$  module via this representation.

**Corollary 8.** As a left  $\mathcal{A}$ -module, the orbit Hilbert space  $\ell_2(\mathcal{S}(x))$  has a Shilov resolution.

*Proof.* This follows from Corollary 4 in which it is shown that  $\rho_{x,\gamma\mu}$  is unitarily equivalent to  $\pi_{x,\gamma}^1$ .  $\square$

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